MATH 245 F20, Exam 3 Solutions

1. Freebie.

2. Let $S = \{x \in \mathbb{Z} : \exists y \in \mathbb{Z},\ x = 6y + 5\}$ and $T = \{x \in \mathbb{Z} : \exists y \in \mathbb{Z},\ x = 2y + 1\}$. Prove or disprove that $S = T$.

The statement is false, and will be disproved with a counterexample. I choose 1; others can work. Note that $1 \in T$, because $2 \cdot 0 + 1 = 1$, and $0 \in \mathbb{Z}$. However, $1 \notin S$, because otherwise we would have an integer y with $1 = 6y+5$, so $-4 = 6y$, and $y = -\frac{2}{3}$ (which is not an integer).

3. Let R, S, T be sets. Prove that $(R \setminus S) \setminus T \subseteq R \setminus (S \setminus T)$.

Let $x \in (R \setminus S) \setminus T$. Then $x \in (R \setminus S) \wedge x \notin T$. By simplification, $x \in (R \setminus S)$, and hence $x \in R \wedge x \notin S$. By simplification again, twice, we get $x \in R$ and $x \notin S$. By addition, $x \notin S \vee x \in T$. By double negation, $\neg x \in S \vee \neg\neg x \in T$. By De Morgan's Law (on propositions), $\neg(x \in S \wedge x \notin T)$, i.e. $\neg(x \in S \setminus T)$, i.e. $x \notin (S \setminus T)$. Going back, we also had $x \in R$. By conjunction, $x \in R \wedge x \notin (S \setminus T)$. Hence $x \in R \setminus (S \setminus T)$.

4. Let $S = \{x\}$. Find a set T that simultaneously satisfies all of the following properties: $S \not\subseteq T$, $2^S \in T$, $2^S \subseteq T$, $S \times 2^S \subseteq T$. Be very careful about notation.

Take $T = \{\underbrace{\{\emptyset, \{x\}\}}_{2^S \in T}, \underbrace{\emptyset, \{x\}}_{2^S \subseteq T}, \underbrace{(x, \emptyset), (x, \{x\})}_{S \times 2^S \subseteq T}, \underbrace{z, \text{your Mom}}_{\text{extras}}\}$.

Correct answers must have the five specific elements listed, as well as perhaps extras (but may not contain element x, since otherwise $S \subseteq T$).

5. Prove or disprove: For all sets S, U with $S \subseteq U$, we have $2^S \cup 2^{(S^c)} = 2^U$.

The statement is false, and needs a counterexample. A correct counterexample consists of three things: a set U, a set S, and an element of 2^U that is not an element of $2^S \cup 2^{(S^c)}$. These last properties must be adequately justified.

Many counterexamples are possible; I choose $U = \{1, 2, 3, 4\}$, $S = \{1, 2\}$, and $x = \{1, 3\}$. Because $x \subseteq U$, in fact $x \in 2^U$. However, $x \not\subseteq S$ so $x \notin 2^S$. Also, $S^c = \{3, 4\}$, so $x \not\subseteq S^c$ and hence $x \notin 2^{(S^c)}$. Since $x \notin 2^S$ and $x \notin 2^{(S^c)}$, by conjunction $x \notin 2^S \wedge x \notin 2^{(S^c)}$. By De Morgan's Law (for propositions), $\neg(x \in 2^S \vee x \in 2^{(S^c)})$ so $\neg x \in (2^S \cup 2^{(S^c)})$ and thus $x \notin (2^S \cup 2^{(S^c)})$.

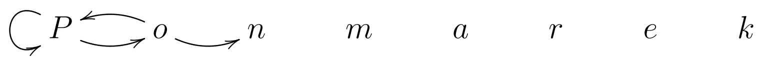
6. Let A, B, C be sets. Prove that $A \times (B \setminus C) = (A \times B) \setminus (A \times C)$.

   Part 1: Let $x \in A \times (B \setminus C)$. Then $x = (m, n)$ with $m \in A$ and $n \in B \setminus C$. Hence $n \in B \wedge n \notin C$; by simplification twice $n \in B$ and $n \notin C$. Hence $x \in A \times B$ since $m \in A$ and $n \in B$. Also $x \notin A \times C$ since $n \notin C$. Hence by conjunction $x \in A \times B \wedge x \notin A \times C$, so $x \in (A \times B) \setminus (A \times C)$.

   Part 2: Let $x \in (A \times B) \setminus (A \times C)$. Then $x \in A \times B \wedge x \notin A \times C$. By simplification twice, $x \in A \times B$ and $x \notin A \times C$. Because $x \in A \times B$, $x = (m, n)$ with $m \in A$ and $n \in B$. Because $x \notin A \times C$ (and yet $m \in A$) we must have $n \notin C$. By conjunction, $n \in B \wedge n \notin C$. Hence $n \in B \setminus C$. Hence $x \in A \times (B \setminus C)$.

7. Let S be the set of letters in your name (choose first or last). Find a relation R on S that is not reflexive, not irreflexive, not symmetric, not antisymmetric, not trichotomous, and not transitive. Give your relation as a directed graph, and fully justify each of these properties.

   Using my last name of Ponomarenko, we have $S = \{P, o, n, m, a, r, e, k\}$. Many relations R can work; here is one example:



   R is not reflexive since $(o, o) \notin R$. R is not irreflexive since $(P, P) \in R$. R is not symmetric since $(o, n) \in R$ but $(n, o) \notin R$. R is not antisymmetric since $(P, o) \in R$ and $(o, P) \in R$ (and $o \neq P$). R is not trichotomous since $(n, m) \notin R$ and $(m, n) \notin R$ (and $m \neq n$). R is not transitive since $(P, o) \in R$ and $(o, n) \in R$ yet $(P, n) \notin R$.

8. Let S be a set, $T \subseteq S$, and R a reflexive relation on S. Prove that $(R|_T)^+$ is reflexive.

   First, note that $(R|_T)^+$ is a relation whose ground set is the same as the ground set of $R|_T$, namely T. Hence we need to prove that $\forall x \in T$, $(x, x) \in (R|_T)^+$. Let $x \in T$ be arbitrary. Since $T \subseteq S$, in fact $x \in S$. Since R is a reflexive relation on S, in fact $(x, x) \in R$. Since $(x, x) \in R$ and $x \in T$, we have $(x, x) \in R|_T = (R|_T)^{(1)}$. Now, $(R|_T)^+ = (R|_T)^{(1)} \cup (R|_T)^{(2)} \cup \cdots$. Since (x, x) is an element of the first set listed, it is an element of the union of all of them, so $(x, x) \in (R|_T)^+$.